ALGEBRO-GEOMETRIC CHARACTERIZATION OF CAYLEY POLYTOPES

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Abstract. In this paper, we give an algebro-geometric characterization of Cayley polytopes. As a special case, we also characterize lattice polytopes with lattice width one by using Seshadri constants.

Contents

1. Introduction

Let $P_0, \ldots, P_r$ be lattice polytopes in $\mathbb{R}^s$. The Cayley sum $P_0 * \cdots * P_r$ is defined to be the convex hull of $(P_0 \times 0) \cup (P_1 \times e_1) \cup \ldots \cup (P_r \times e_r)$ in $\mathbb{R}^s \times \mathbb{R}^r$ for the standard basis $e_1, \ldots, e_r$ of $\mathbb{R}^r$.

A lattice polytope $P \subset \mathbb{R}^n$ is said to be a Cayley polytope of length $r+1$, if there exists an affine isomorphism $\mathbb{Z}^n \cong \mathbb{Z}^{n-r} \times \mathbb{Z}^r$ identifying P with the Cayley sum $P_0 * \cdots * P_r$ for some lattice polytopes $P_0, \ldots, P_r$ in $\mathbb{R}^{n-r}$. In other words, P is a Cayley polytope of length $r+1$ if and only if P is mapped onto a unimodular r-simplex by a lattice projection $\mathbb{R}^n \to \mathbb{R}^r$.

Cayley polytopes are related to discriminants, resultants, and dual defects. See for instance [DDP], [DN], [GKZ].

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On the other hand, a polarized toric variety (X_P, L_P) is defined for any lattice polytope $P \subset \mathbb{R}^n$ of dimension n.

In this paper, we give an algebro-geometric characterization of Cayley polytopes:

Theorem 1.1. *Let $P \subset \mathbb{R}^n$ be a lattice polytope of dimension n. Then P is a Cayley polytope of length $r+1$ if and only if (X_P, L_P) is covered by r-planes.*

See Definition 2.4 for the definition of "covered by r-planes". An important point to note here is that we do not need any assumption on the singularities of X_P nor the lattice spanned by $P \cap \mathbb{Z}^n$. As a corollary of this theorem, we obtain a sufficient condition such that a lattice polytope P is a Cayley polytope by using dual defects (see Corollary 4.2). In [CD], projective $\mathbb{Q}$-factorial toric varieties covered by lines (=1-planes) are studied in detail, and Theorem 1.1 and Corollary 4.2 generalize some of their results.

We investigate the case $r = 1$ a little more. For a polarized variety (X, L), we can define a positive number $\varepsilon(X, L; 1)$, which is called the Seshadri constant of (X, L) at a very general point. This is an invariant measuring the positivity of (X, L). In the following theorem, we characterize Cayley polytopes of length 2 by using Seshadri constants:

Theorem 1.2. *Let $P \subset \mathbb{R}^n$ be a lattice polytope of dimension n. Then the following are equivalent:*

i) *P is a Cayley polytope of length 2,*
ii) *(X_P, L_P) is covered by lines,*
iii) *$\varepsilon(X_P, L_P; 1) = 1$.*

In general, it is very difficult to compute Seshadri constants. Theorem 1.2 gives an explicit description for which lattice polytope P the Seshadri constant $\varepsilon(X_P, L_P; 1)$ is one.

This paper is organized as follows: In Section 2, we make some preliminaries. In Section 3, we give the proof of Theorem 1.1. In Section 4, we state a relation of Cayley polytopes and dual defects. In Section 5, we prove Theorem 1.2.

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2. Preliminaries

We denote by $\mathbb{N}, \mathbb{Z}, \mathbb{R}$, and $\mathbb{C}$ the set of all natural numbers, integers, real numbers, and complex numbers respectively. In this paper, $\mathbb{N}$ contains 0. We denote $\mathbb{C} \setminus 0$ by $\mathbb{C}^{\times}$.

Let us denote by $e_1, \dots, e_n$ the standard basis of $\mathbb{Z}^n$ or $\mathbb{R}^n$. A lattice polytope in $\mathbb{R}^n$ is the convex hull of finite points in $\mathbb{Z}^n$. The dimension of a lattice polytope $P \subset \mathbb{R}^n$ is the dimension of the affine space spanned by P. For a subset S in an $\mathbb{R}$-vector space, we write $\Sigma(S)$ for the closed convex cone spanned by S.

Throughout this paper, we consider varieties or schemes over the complex number field $\mathbb{C}$. For a variety X, we say a property holds at a general point of X if it holds for all points in the complement of a proper algebraic subset. For a variety X, we say a property holds at a very general point of X if it holds for all points in the complement of the union of countably many proper subvarieties.

2.1. **Cayley polytopes and r-planes.**

We say a linear map $\mathbb{R}^n \to \mathbb{R}^r$ is a lattice projection if it is induced from a surjective group homomorphism $\mathbb{Z}^n \to \mathbb{Z}^r$.

Let us recall the definitions of Cayley polytopes and lattice width.

Definition 2.1. Let P be a lattice polytope in $\mathbb{R}^n$ and r a positive integer. We say P is a Cayley polytope of length $r+1$ if there exists a lattice projection onto a unimodular r-simplex. A unimodular r-simplex is a lattice polytope in $\mathbb{R}^r$ which is identified with the convex hull of $0, e_1, \dots, e_r$ by a $\mathbb{Z}$-affine translation.

Definition 2.2. Let P be a lattice polytope in $\mathbb{R}^n$. The lattice width of P is the minimum of $\max_{u \in P}\langle u, v\rangle - \min_{u \in P}\langle u, v\rangle$ over all non-zero integer linear forms v.

Remark 2.3. See [BN] or [DHNP] for other definitions of Cayley polytopes. Note that P has lattice width one if and only if P is an n-dimensional Cayley polytope of length 2.

We define r-planes as follows:

Definition 2.4. Let r be a positive integer. A polarized variety (X, L) is called an r-plane if it is isomorphic to $(\mathbb{P}^r, \mathcal{O}(1))$ as a polarized variety. Sometimes we say X is an r-plane if the polarization L is clear (e.g. a subvariety in a polarized variety).

Let (X, L) be a polarized variety. We say that (X, L) is covered by r-planes, if for any general point $p \in X$ there exists an r-plane $Z \subset X$ containing p. When $r = 1$, we say that (X, L) is covered by lines.

2.2. **Toric varieties.** In this subsection, we prepare notations about toric varieties used in this paper. We refer the reader to [Fu] for a further treatment.

Let P be a lattice polytope of dimension n in $\mathbb{R}^n$. Then we can define the polarized toric variety associated to P as

$$(X_P, L_P) = (\operatorname{Proj} \mathbb{C}[\Gamma_P], \mathcal{O}(1)),$$

where $\Gamma_P := \Sigma(\{1\} \times P) \cap (\mathbb{N} \times \mathbb{Z}^n)$ is a subsemigroup of $\mathbb{N} \times \mathbb{Z}^n$. We consider that Γ_P is graded by $\mathbb{N}$, whose degree k part is $\Gamma_P \cap (\{k\} \times \mathbb{Z}^n) = \{k\} \times (kP \cap \mathbb{Z}^n)$. There exists a natural action on X_P by the torus $(\mathbb{C}^\times)^n$. We denote the maximal orbit in X_P by $O_P = (\mathbb{C}^\times)^n$. The action $(\mathbb{C}^\times)^n \times X_P \to X_P$ is the extension of the group structure

$$(\mathbb{C}^\times)^n \times (\mathbb{C}^\times)^n \to (\mathbb{C}^\times)^n; (a, b) \mapsto (a_1 b_1, \ldots, a_n b_n),$$

where $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in (\mathbb{C}^\times)^n$.

By definition, a lattice point u in $P \cap \mathbb{Z}^n$ corresponds to a global section x^u in $H^0(X_P, L_P)$. It is well known that such global sections form a basis of $H^0(X_P, L_P)$ and the linear system $|L_P|$ is base point free. We denote by ϕ_P the morphism $X_P \to \mathbb{P}^N$ defined by $|L_P|$, where $N = \#(P \cap \mathbb{Z}^n) - 1$. Note that ϕ_P is a finite morphism onto the image.

We will use the following lemma repeatedly in the subsequent sections. This is well known, but we prove it for the convenience of the reader:

Lemma 2.5. *Let $P \subset \mathbb{R}^n$ be a lattice polytope of dimension n and $\pi : \mathbb{R}^n \to \mathbb{R}^r$ a lattice projection. Then there is a birational finite morphism onto the image $\iota : X_{\pi(P)} \to X_P$ such that $\iota^*(L_P) = L_{\pi(P)}$.*

Proof. Consider the following diagram:

$$\begin{tikzcd}
\Sigma(\{1\} \times P) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{R} \times \mathbb{R}^n \arrow[d, "\mathrm{id}_{\mathbb{R}} \times \pi"] \\
\Sigma(\{1\} \times \pi(P)) \arrow[r, hook] & \mathbb{R} \times \mathbb{R}^r.
\end{tikzcd}$$

By intersecting with $\mathbb{N} \times \mathbb{Z}^n$ or $\mathbb{N} \times \mathbb{Z}^r$, we have

$$\begin{tikzcd}
\Gamma_P = \Sigma(\{1\} \times P) \cap (\mathbb{N} \times \mathbb{Z}^n) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{N} \times \mathbb{Z}^n \arrow[d, "\mathrm{id}_{\mathbb{N}} \times \pi|_{\mathbb{Z}^n}"] \\
\Gamma_{\pi(P)} = \Sigma(\{1\} \times \pi(P)) \cap (\mathbb{N} \times \mathbb{Z}^r) \arrow[r, hook] & \mathbb{N} \times \mathbb{Z}^r.
\end{tikzcd}$$

Set $\Gamma' = (\mathrm{id}_{\mathbb{N}} \times \pi|_{\mathbb{Z}^r})(\Gamma_P)$. Then the above diagram induces

$$\begin{tikzcd} X_P = \operatorname{Proj} \mathbb{C}[\Gamma_P] & (\mathbb{C}^\times)^n \arrow[l, hook'] \\ \operatorname{Proj} \mathbb{C}[\Gamma'] \arrow[u, hook] \arrow[ur, phantom, "\circlearrowleft"] & (\mathbb{C}^\times)^r. \arrow[l, hook'] \arrow[u, hook] \end{tikzcd}$$

Note that Γ' generates $\mathbb{Z} \times \mathbb{Z}^r$ as a group and $\Sigma(\Gamma') = \Sigma(\{1\} \times \pi(P))$ because P is n-dimensional. So there exists the normalization morphism (cf. [Ei, Exercise 4.22])

$$\iota : X_{\pi(P)} \to \operatorname{Proj} \mathbb{C}[\Gamma'] \ (\hookrightarrow X_P).$$

By the construction of ι, it is clear that $\iota^*(L_P) = L_{\pi(P)}$. $\square$

3. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1=Theorem 3.1:

Theorem 3.1. *Let $P \subset \mathbb{R}^n$ be a lattice polytope of dimension n. Then P is a Cayley polytope of length $r+1$ if and only if (X_P, L_P) is covered by r-planes.*

Proof. If P is a Cayley polytope of length $r+1$, then there exists a birational finite morphism onto the image $\iota : \mathbb{P}^r \to X_P$ by Lemma 2.5. Furthermore, there exists a finite morphism $\phi = \phi_P : X_P \to \mathbb{P}^N$. We denote by $Z \subset X_P$ the image of $\mathbb{P}^r$ by the morphism ι. Set $Z' = \phi \circ \iota(\mathbb{P}^r) = \phi(Z)$ in $\mathbb{P}^N$. Since $(\phi \circ \iota)^* \mathcal{O}_{\mathbb{P}^N}(1) = \iota^* L_P = \mathcal{O}_{\mathbb{P}^r}(1)$, it holds that

$$1 = \mathcal{O}_{\mathbb{P}^r}(1)^r = \deg(\phi \circ \iota) \cdot \mathcal{O}_{Z'}(1)^r,$$

where $\deg(\phi \circ \iota)$ is the degree of the finite morphism $\phi \circ \iota : \mathbb{P}^r \to Z'$. Both of $\deg(\phi \circ \iota)$ and $\mathcal{O}_{Z'}(1)^r$ are positive integers. Hence we have $\deg(\phi \circ \iota) = \mathcal{O}_{Z'}(1)^r = 1$. Thus $Z' \subset \mathbb{P}^N$ is an r-plane and $\phi \circ \iota$ is birational. Since Z' is smooth and $\phi \circ \iota$ is a birational finite morphism, $\phi \circ \iota : \mathbb{P}^r \to Z'$ is an isomorphism. Hence $\iota : (\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)) \to (Z, L_P|_Z)$ is also an isomorphism, i.e. Z is an r-plane in X_P. Note that $Z \cap O_P$ is not empty, so by the torus action (X_P, L_P) is covered by r-planes.

Let us prove the converse, so assume that (X_P, L_P) is covered by r-planes. We may assume that $0 \in \mathbb{Z}^n$ is a vertex of P. Thus throughout this proof, set $P \cap \mathbb{Z}^n = \{u_0, \ldots, u_N\}$ and $u_0 = 0$.

Since (X_P, L_P) is covered by r-planes, there exists an r-plane $Z \subset X_P$ containing $1_P := (1, \ldots, 1) \in (\mathbb{C}^\times)^n = O_P \subset X_P$. Let $Z' \subset \mathbb{P}^N$ be the image of Z by $\phi = \phi_P : X_P \to \mathbb{P}^N$. Then it is easy to see that $\phi : (Z, L_P|_Z) \to (Z', \mathcal{O}(1))$ is an isomorphism and $Z' \subset \mathbb{P}^N$ is an r-plane by the argument similar to that of the "only if" part of this

proof. Note that Z' contains $1_N := (1, \dots, 1) \in (\mathbb{C}^\times)^N \subset \mathbb{P}^N$ because $\phi(1_P) = 1_N$.

Our idea of proof is the following:

Step 1. *By using the torus action, we degenerate Z to another r-plane $\widetilde{Z} \subset X_P$ containing 1_P such that the embedding $\mathbb{P}^r \cong \phi(\widetilde{Z}) \hookrightarrow \mathbb{P}^N$ is a toric morphism, i.e. a morphism induced by a lattice projection $\mathbb{R}^N \to \mathbb{R}^r$ as Lemma 2.5.*

Step 2. *By using the above lattice projection, we define another lattice projection $\mathbb{R}^n \to \mathbb{R}^r$ which maps P onto a unimodular r-simplex.*

To clarify the idea of proof, we first show the case $r = 1$. When $r = 1$, we write C, l instead of Z, Z'.

Step 1. By definition, l is a line on $\mathbb{P}^N$ containing 1_N. Thus we can write

$$l \cap \mathbb{C}^N = 1_N + \mathbb{C}a$$

for some vector $a = (a_1, \dots, a_N) \in \mathbb{C}^N \setminus 0$, where $\mathbb{C}^N \subset \mathbb{P}^N$ is the open set defined by $T_0 \neq 0$ for the homogeneous coordinates $T_0, \dots, T_N$. For any point p in O_P, there exist automorphisms $p^{-1}\cdot : X_P \to X_P$ and $\phi(p)^{-1}\cdot : \mathbb{P}^N \to \mathbb{P}^N$ induced by torus actions. Hence any point p in $C \cap O_P$ induces an isomorphism

$$\phi|_{p^{-1}\cdot C} : p^{-1} \cdot C \to \phi(p)^{-1} \cdot l.$$

Since $1_P = p^{-1} \cdot p$ is contained in $p^{-1} \cdot C$, the line $\phi(p)^{-1} \cdot l$ contains 1_N. From this, we have

$$(\phi(p)^{-1} \cdot l) \cap \mathbb{C}^N = 1_N + \mathbb{C}\ \phi(p)^{-1} \cdot a.$$

Let us denote $\phi(p) = 1_N + t_p a \in l \cap (\mathbb{C}^\times)^N$ for $t_p \in \mathbb{C}$. Moving $p \in C \cap O_P$ so that $|t_p| \to +\infty$ and taking limits, we have a morphism

$$\phi|_{\widetilde{C}} : \widetilde{C} \to \tilde{l},$$

where $\widetilde{C} \subset X_P$ and $\tilde{l} \subset \mathbb{P}^N$ are the limits of $p^{-1} \cdot C$ and $\phi(p)^{-1} \cdot l$ in the Hilbert schemes respectively. A limit of lines is also a line, so $\phi : (\widetilde{C}, L_P|_{\widetilde{C}}) \to (\tilde{l}, \mathcal{O}(1)) = (\mathbb{P}^1, \mathcal{O}(1))$ is also isomorphic. Moreover $p^{-1} \cdot C$ contains 1_P for any p, so does $\widetilde{C}$.

When $|t_p| \to +\infty$,

$$\phi(p)^{-1} \cdot t_p a = ((1 + t_p a_1)^{-1} t_p a_1, \dots, (1 + t_p a_N)^{-1} t_p a_N)$$

converges to $\tilde{a} = (\tilde{a}_1, \dots, \tilde{a}_N) \in \mathbb{C}^N$, where

$$\tilde{a}_j = \begin{cases} 1 & \text{if } a_j \neq 0 \\ 0 & \text{if } a_j = 0 \end{cases}.$$

Since $\tilde{l} \cap \mathbb{C}^N$ is the limit of $1_N + \mathbb{C}\ \phi(p)^{-1} \cdot a = 1_N + \mathbb{C}\ \phi(p)^{-1} \cdot t_p a$, we can write

$$\tilde{l} \cap \mathbb{C}^N = 1_N + \mathbb{C}\tilde{a}.$$

From this description, $\mathbb{C}^\times = \tilde{l} \cap (\mathbb{C}^\times)^N \hookrightarrow (\mathbb{C}^\times)^N \subset \mathbb{P}^N$ can be written as

$$t \mapsto (t^{\tilde{a}_1}, \ldots, t^{\tilde{a}_N})$$

for $t \in \mathbb{C}^\times$. Therefore $\mathbb{P}^1 = \tilde{l} \hookrightarrow \mathbb{P}^N$ is the toric morphism defined by the lattice projection

$$\mathbb{R}^N \to \mathbb{R}; e_j \mapsto \tilde{a}_j$$

and the N-simplex in $\mathbb{R}^N$ spanned by $0, e_1, \ldots, e_N$ as Lemma 2.5.

Step 2. Restricting the diagram

$$\begin{tikzcd} X_P \arrow[r, "\phi"] & \mathbb{P}^N \\ & \widetilde{C} \cong \tilde{l} = \mathbb{P}^1 \arrow[ul, hook, "\alpha"] \arrow[u, hook, "\beta"'] \end{tikzcd}$$

to the maximal orbits, we have

$$\begin{tikzcd} (\mathbb{C}^\times)^n \arrow[r, "\phi"] & (\mathbb{C}^\times)^N \\ & \mathbb{C}^\times. \arrow[ul, hook, "\alpha"] \arrow[u, hook, "\beta"'] \end{tikzcd}$$

Considering the coordinate rings, we have

$$\begin{tikzcd} \mathbb{C}[\mathbb{Z}^n] \arrow[dr, "f"'] & \mathbb{C}[\mathbb{Z}^N] \arrow[l, "\phi^*"'] \arrow[d, "g"] \\ & \mathbb{C}[\mathbb{Z}] \end{tikzcd}$$

By the definition of ϕ and the assumption $u_0 = 0$, the ring homomorphism ϕ^* is induced by the following group homomorphism

$$\pi : \mathbb{Z}^N \to \mathbb{Z}^n; e_j \mapsto u_j$$

for $1 \leq j \leq N$, i.e.

$$\phi^*(x^{u'}) = x^{\pi(u')}$$

holds for each $u' \in \mathbb{Z}^N$. On a while, g is induced by the surjective group homomorphism

$$\mu : \mathbb{Z}^N \to \mathbb{Z}; e_j \mapsto \tilde{a}_j$$

from Step 1. By using π and μ, we can define a group homomorphism $\pi' : \mathbb{Z}^n \to \mathbb{Z}$ which induces the ring homomorphism f as follows:

For any $u \in \mathbb{Z}^n$, there exists a positive integer m such that mu is contained in $\pi(\mathbb{Z}^N)$. Thus we can take $u' \in \mathbb{Z}^N$ such that $mu = \pi(u')$. By the commutativity of the above diagram, we have $f(x^{mu}) = g(x^{u'}) = x^{\mu(u')}$. Since $f(x^{mu}) = f(x^u)^m$ holds, $\mu(u')/m$ must be contained in $\mathbb{Z}$ and $f(x^u) = c \cdot x^{\mu(u')/m}$ for some $c \in \mathbb{C}^\times$. Furthermore $1 \in \mathbb{C}^\times$ is mapped to $1_P \in X_P$ by α, so c must be 1. Thus we can define $\pi' : \mathbb{Z}^n \to \mathbb{Z}$ by

$$\pi'(u) = \frac{\mu(u')}{m}.$$

It is easy to show that π' is well defined and a group homomorphism which induces f.

Since $\alpha : \mathbb{P}^1 \to X_P$ is a closed embedding, f is surjective. This means that π' is also surjective. Furthermore $\mu = \pi' \circ \pi$ holds by the definition of π', hence we have $\pi'(u_j) = \pi' \circ \pi(e_j) = \mu(e_j) = \tilde{a}_j \in \{0, 1\}$ for each $j \in \{1, \ldots, N\}$. Since P is the convex hull of $0, u_1, \ldots, u_N$, the lattice projection induced by π' maps P onto the closed interval $[0, 1]$ in $\mathbb{R}$. This means that P is a Cayley polytope of length 2.

When $r > 1$, the idea is same. Let $Z \subset X_P$ be an r-plane containing 1_P and $Z' = \phi(Z) \subset \mathbb{P}^N$. Since Z' is an r-plane in $\mathbb{P}^N$ containing 1_N, we can write

$$Z' \cap \mathbb{C}^N = 1_N + V,$$

where V is an r-dimensional linear subspace of $\mathbb{C}^N$. Then we can choose a basis $a_1, \ldots, a_r \in \mathbb{C}^n$ of V such that $j_1 > j_2 > \ldots > j_r \cdots (*)$ holds, where $a_i = (a_{i1}, \ldots, a_{iN})$ and $j_i = \min\{\, j \mid a_{ij} \neq 0\}$. In other words, the matrix ${}^t({}^t a_1, \ldots, {}^t a_r)$ is the following type:

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_r \end{pmatrix} = \begin{pmatrix} 0 & * & \ldots & & & & & \\ 0 & 0 & 0 & * & \ldots & & & \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & \ldots & 0 & * & \ldots \end{pmatrix}.$$

Similarly to the case $r = 1$, any point $p \in O_P$ induces an isomorphism

$$\phi|_{p^{-1}\cdot Z} : p^{-1} \cdot Z \to \phi(p)^{-1} \cdot Z'.$$

Choose $p \in Z \cap O_P$ satisfying $\phi(p) = 1_N + t_p a_r$ for some $t_p \in \mathbb{C}$ and take limits of $p^{-1} \cdot Z$ and $\phi(p)^{-1} \cdot Z'$ by moving p so that $|t_p| \to +\infty$. We denote the limits by $Z^{(1)} \subset X_P$ and $Z'^{(1)} \subset \mathbb{P}^N$ respectively, both of which are r-planes.

Then

$$\begin{aligned}(\phi(p)^{-1}\cdot Z')\cap\mathbb{C}^N &= 1_N+\phi(p)^{-1}\cdot V\\ &= 1_N+\bigoplus_{i=1}^{r}\mathbb{C}\ \phi(p)^{-1}\cdot a_i\end{aligned}$$

holds because $1_N=\phi(p)^{-1}\cdot\phi(p)$ is contained in $\phi(p)^{-1}\cdot Z'$. For $t_p\neq 0$, it holds that

$$\begin{aligned}\mathbb{C}\ \phi(p)^{-1}\cdot a_i &= \mathbb{C}((1+t_pa_{r1})^{-1}a_{i1},\dots,(1+t_pa_{rN})^{-1}a_{iN})\\ &= \mathbb{C}((1+t_pa_{r1})^{-1}t_pa_{i1},\dots,(1+t_pa_{rN})^{-1}t_pa_{iN}).\end{aligned}$$

When $|t_p|\to+\infty$,

$$\phi(p)^{-1}\cdot a_i=((1+t_pa_{r1})^{-1}a_{i1},\dots,(1+t_pa_{rN})^{-1}a_{iN})$$

converges to $a_i^{(1)}=(a_{ij}^{(1)})_j$ for $i\neq r$, where

$$a_{ij}^{(1)}=\begin{cases}a_{ij} & \text{if } a_{rj}=0\\ 0 & \text{if } a_{rj}\neq 0\ .\end{cases}$$

Note that $a_i^{(1)}$ is not 0 since $a_{ij_i}\neq 0$ and $a_{rj_i}=0$ for any $i\neq r$ by $(*)$.

On the other hand,

$$\phi(p)^{-1}\cdot t_pa_r=((1+t_pa_{r1})^{-1}t_pa_{r1},\dots,(1+t_pa_{rN})^{-1}t_pa_{rN})$$

converges to $a_r^{(1)}=(a_{rj}^{(1)})_j$, where

$$a_{rj}^{(1)}=\begin{cases}0 & \text{if } a_{rj}=0\\ 1 & \text{if } a_{rj}\neq 0\ .\end{cases}$$

It is easy to see that $a_1^{(1)},\dots,a_N^{(1)}$ are linearly independent by the condition $(*)$, so we have

$$Z'^{(1)}\cap\mathbb{C}^N=1_N+\bigoplus_{i=1}^{r}\mathbb{C}a_i^{(1)}.$$

Now $Z^{(1)}$ is an r-plane containing 1_P and $\phi|_{Z^{(1)}}:Z^{(1)}\to Z'^{(1)}$ is also isomorphic. We choose $p\in Z^{(1)}\cap O_P$ satisfying $\phi(p)=1_N+t_pa_{r-1}^{(1)}$ for some t_p and take limits by moving p so that $|t_p|\to+\infty$. We denote the limits by $Z^{(2)}$ and $Z'^{(2)}$ respectively. Note that $a_1^{(1)},\dots,a_N^{(1)}$ also satisfy the condition $(*)$. In fact $\min\{\,j\mid a_{ij}\neq 0\}=\min\{\,j\mid a_{ij}^{(1)}\neq 0\}$ holds by definition. By similar arguments, we have

$$Z'^{(2)}\cap\mathbb{C}^N=1_N+\bigoplus_{i=1}^{r}\mathbb{C}a_i^{(2)},$$

where $a_i^{(2)} = (a_{ij}^{(2)})_j$ and

$$a_{ij}^{(2)} = \begin{cases} a_{ij}^{(1)} & \text{if } a_{r-1,j}^{(1)} = 0 \\ 0 & \text{if } a_{r-1,j}^{(1)} \neq 0 \end{cases}$$

for $i \neq r-1$, and

$$a_{r-1,j}^{(2)} = \begin{cases} 0 & \text{if } a_{r-1,j}^{(1)} = 0 \\ 1 & \text{if } a_{r-1,j}^{(1)} \neq 0 \; . \end{cases}$$

Note that $a_{rj}^{(1)} = 0$ if $a_{r-1,j}^{(1)} \neq 0$, so $a_r^{(2)} = a_r^{(1)}$ holds.

By repeating these operations r times, we obtain $Z^{(r)} \subset X_P$ and $Z'^{(r)} \subset \mathbb{P}^N$ containing 1_P and 1_N respectively, and linearly independent vectors $a_1^{(r)}, \ldots, a_r^{(r)} \in \mathbb{C}^N$ such that

i) $\phi|_{Z^{(r)}} : (Z^{(r)}, L_P|_{Z^{(r)}}) \to (Z'^{(r)}, \mathcal{O}(1))$ is an isomorphism,

ii) $Z'^{(r)} \cap \mathbb{C}^N = 1_N + \bigoplus_{i=1}^r \mathbb{C} a_i^{(r)}$,

iii) $a_{ij}^{(r)} = 0$ or 1. For each j, there are at most one i such that $a_{ij}^{(r)} = 1$.

Note that $Z^{(r)}$ and $Z'^{(r)}$ are r-planes by i) and ii). By iii), we can define $i_j \in \{0, 1, \ldots, r\}$ for $j = 1, \ldots, N$ as follows:
If $\{\, i \mid a_{ij}^{(r)} = 1\}$ is not empty, we set $i_j = i$ satisfying $a_{ij}^{(r)} = 1$. Otherwise we set $i_j = 0$.

As in the case $r = 1$, we have the following diagram

$$\begin{tikzcd} \mathbb{C}[\mathbb{Z}^n] \arrow[dr, "f"'] & \mathbb{C}[\mathbb{Z}^N] \arrow[l, "\phi^*"'] \arrow[d, "g"] \\ & \mathbb{C}[\mathbb{Z}^r] \end{tikzcd}$$

induced from

$$\begin{tikzcd} X_P \arrow[r, "\phi"] & \mathbb{P}^N \\ & Z^{(r)} \cong Z'^{(r)} = \mathbb{P}^r. \arrow[ul, hook, "\alpha"] \arrow[u, hook, "\beta"'] \end{tikzcd}$$

From the construction of $Z'^{(r)}$ and e_{i_j}, it is easy to see that g is induced by the group homomorphism

$$\mu : \mathbb{Z}^N \to \mathbb{Z}^r ; e_j \mapsto e_{i_j},$$

where we consider e_0 as $0 \in \mathbb{Z}^r$. Similar to the case $r = 1$, we can define a surjective group homomorphism $\pi' : \mathbb{Z}^n \to \mathbb{Z}^r$ such that f is induced by π' and $\mu = \pi' \circ \pi$, where $\pi : \mathbb{Z}^N \to \mathbb{Z}^n$ is the group homomorphism

which induces ϕ^*. The surjectivity of π' follows from the that of f. It is easily shown that the lattice projection induced from π' maps P onto the r-simplex spanned by $0, e_1, \dots, e_r \in \mathbb{R}^r$ because $\pi'(u_j) = e_{i_j}$ holds for each j. Thus P is a Cayley polytope of length $r+1$. □

4. Dual defects

Cayley polytopes are often studied with related to dual defects.

Definition 4.1. Let $X \subset \mathbb{P}^N$ be a projective variety. The dual variety X^* of X is the closure of all points $H \in (\mathbb{P}^N)^\vee$ such that as a hyperplane H contains the tangent space $T_{X,p}$ for some smooth point $p \in X$, where $(\mathbb{P}^N)^\vee$ is the dual projective space. A variety X in $\mathbb{P}^N$ is said to be dual defective if the dimension of X^* is less than $N-1$. The dual defect of X is the natural number $N - 1 - \dim X^*$.

As an easy corollary of Theorem 1.1, we obtain a sufficient condition such that P is a Cayley polytope by using dual defects. This is a generalization of a result proved in [CC] or [Es], which is the case $r = 1$ of the following:

Corollary 4.2. *Let P be a lattice polytope of dimension n in $\mathbb{R}^n$. Assume that the lattice points in P span the lattice $\mathbb{Z}^n$, and the dual defect of the image $\phi_P(X_P) \subset \mathbb{P}^N$ is a positive integer r. Then P is a Cayley polytope of length $r+1$. In particular, P has lattice width one.*

Proof. It is well known that a projective variety $X \subset \mathbb{P}^N$ has dual defect r, then X is covered by r-planes (cf. [Te, Theorem 1.18] for example). Thus if the dual defect of $\phi_P(X_P)$ is r, then $\phi_P(X_P)$ is covered by r-planes. The assumption that $P \cap \mathbb{Z}$ spans the lattice $\mathbb{Z}^n$ means that $\phi_P : X_P \to \phi_P(X_P)$ is birational. Since $\phi_P(X_P)$ is covered by r-planes and $\phi_P : X_P \to \phi_P(X_P)$ is a birational finite morphism, (X_P, L_P) is also covered by r-planes. Therefore P is a Cayley polytope of length $r+1$ by Theorem 1.1. □

Remark 4.3. (1) In Corollary 4.2, the assumption that $P \cap \mathbb{Z}$ spans $\mathbb{Z}^n$ is necessary. For example, let $P \subset \mathbb{R}^3$ be the convex hull of $(0,0,0), (1,1,0), (1,0,1)$ and $(0,1,1)$. Then the image $\phi_P(X_P) \subset \mathbb{P}^3$ is $\mathbb{P}^3$, so the dual defect of $\phi_P(X_P)$ is 3. But P is not a Cayley polytope of length 4.
(2) The converse of Corollary 4.2 does not hold. For example, let $P \subset \mathbb{R}^2$ be the convex hull of $(0,0), (1,0), (0,1), (1,1)$. Then P is a Cayley polytope of length 2, but $\phi_P(X_P) = X_P \subset \mathbb{P}^3$ is a smooth quadric surface, which is not dual defective.
(3) There exists an explicit description of the dual defectivity for (X_P, L_P)

if X_P is smooth [DN]. But in singular cases, dual defectivities of toric varieties are not so well known.

5. Lattice width one and Seshadri constants

In this section, we characterize lattice polytopes with lattice width one by using Seshadri constants. Chapter 5 of [La] is a good reference of Seshadri constants.

Definition 5.1. Let (X, L) be a polarized variety of dimension n and p a point in X. The Seshadri constant $\varepsilon(X, L; p)$ of L at p is defined to be

$$\varepsilon(X, L; p) = \inf_{C} \frac{C.L}{m_p(C)},$$

where C moves all curves on X containing p and $m_p(C)$ is the multiplicity of C at p. It is easily shown that

$$\varepsilon(X, L; p) = \max\{s > 0 \mid \mu^* L - sE \text{ is nef }\},$$

where μ is the blowing up at p and E is the exceptional divisor.

It is well known that $\varepsilon(X, L; p)$ is constant for very general p, so we can define $\varepsilon(X, L; 1)$ to be

$$\varepsilon(X, L; 1) = \varepsilon(X, L; p)$$

for very general $p \in X$.

Seshadri constants are invariants which measure the positivity of ample line bundles. Computations of Seshadri constants sometimes give interesting geometric consequences, but it is very difficult to compute Seshadri constants in general. However, if $|L|$ is base point free, we know whether $\varepsilon(X, L; 1) = 1$ or not by considering lines on (X, L):

Proposition 5.2. *Let (X, L) be a polarized variety and assume that the linear system $|L|$ is base point free. Then $\varepsilon(X, L; 1) = 1$ if and only if (X, L) is covered by lines.*

Proof. "If" part is easy. In fact if (X, L) is covered by lines, then clearly $\varepsilon(X, L; 1) \leq 1$ holds by the definition of Seshadri constants. On the other hand, it is well known that $\varepsilon(X, L; 1) \geq 1$ holds for any base point free L [La, Example 5.1.18]. So $\varepsilon(X, L; 1) = 1$ holds.

Thus it is enough to show the "only if" part. Assume that $\varepsilon(X, L; 1) = 1$ holds. Let $\phi : X \to \mathbb{P}^N$ be the morphism defined by $|L|$, and set d be the degree of the finite morphism $\phi : X \to \phi(X)$. Fix a very general point $p \in X$ and set $q = \phi(p) \in \mathbb{P}^N$. Then $\phi^{-1}(q) = \{p_1, \ldots, p_d\}$ is a set of d points in X, where $p = p_1$.

We consider the following diagram:

$$\begin{tikzcd}
 & \widetilde{X} \arrow[dl, "\pi"'] \arrow[d, "\widetilde{\phi}"'] \arrow[ddr, bend left, "f"] & \\
X \arrow[d, "\phi"'] & \widetilde{\mathbb{P}}^N \arrow[dl, "\pi'"] \arrow[dr, "f'"'] & \\
\mathbb{P}^N & & \mathbb{P}^{N-1}.
\end{tikzcd}$$

In the above diagram, $\pi : \widetilde{X} \to X$ and $\pi' : \widetilde{\mathbb{P}}^N \to \mathbb{P}^N$ are the blowing ups along $\{p_1, \dots, p_d\}$ and q respectively. Then ϕ, π, and π' induce a finite morphism $\widetilde{\phi} : \widetilde{X} \to \widetilde{\mathbb{P}}^N$. Let $E_1, \dots, E_d$ and E be the exceptional divisors over $\{p_1, \dots, p_d\}$ and q respectively. Since $\pi'^*\mathcal{O}_{\mathbb{P}^N}(1) - E$ is base point free, this induces a morphism $f' : \widetilde{\mathbb{P}}^N \to \mathbb{P}^{N-1}$. Set $f = f' \circ \widetilde{\phi} : \widetilde{X} \to \mathbb{P}^{N-1}$. Note that $f^*\mathcal{O}_{\mathbb{P}^{N-1}}(1) = \pi^*L - \sum_{i=1}^d E_i$ and f is the morphism induced by $|\pi^*L - \sum_{i=1}^d E_i|$.

By the assumption that $\varepsilon(X, L; 1) = 1$ and p is very general, $\pi^*L - E_1$ is nef but not ample. So there exists a subvariety $\widetilde{Z}$ in $\widetilde{X}$ such that $\widetilde{Z}.(\pi^*L - E_1)^{\dim \widetilde{Z}} = 0$ (see [La, Proposition 5.1.9]). Furthermore, $\widetilde{Z}.(\pi^*L - \sum_{i=1}^d E_i)^{\dim \widetilde{Z}} \geq 0$ by the freeness of $\pi^*L - \sum_{i=1}^d E_i$. Hence

$$\begin{aligned}
0 &\leq \widetilde{Z}.(\pi^*L - \sum_{i=1}^d E_i)^{\dim \widetilde{Z}} \\
&= Z.L^{\dim Z} - \sum_{i=1}^d m_{p_i}(Z) \\
&\leq Z.L^{\dim Z} - m_{p_1}(Z) \\
&= \widetilde{Z}.(\pi^*L - E_1)^{\dim \widetilde{Z}} = 0,
\end{aligned}$$

where $Z = \pi(\widetilde{Z})$ is the image of $\widetilde{Z}$ by π and $m_{p_i}(Z)$ is the multiplicity of Z at p_i. Thus we obtain

$$0 = \widetilde{Z}.(\pi^*L - \sum_{i=1}^d E_i)^{\dim \widetilde{Z}} = \widetilde{Z}.(\pi^*L - E_1)^{\dim \widetilde{Z}}$$

and $m_{p_i}(Z) = 0$ for $i \neq 1$. This means $p_i \notin Z$, or equivalently $\widetilde{Z} \cap E_i = \emptyset$ for $i \neq 1$. The equality $\widetilde{Z}.(\pi^*L - \sum_{i=1}^d E_i)^{\dim \widetilde{Z}} = 0$ implies $\dim f(\widetilde{Z}) < \dim \widetilde{Z}$. So there exists a curve $\widetilde{C} \subset \widetilde{Z}$ such that $f(\widetilde{C})$ is a point. Set $C = \pi(\widetilde{C})$ be the image of $\widetilde{C}$ in X. Since the morphism $\widetilde{\phi}$ is finite onto the image, $\widetilde{\phi}(\widetilde{C})$ is a curve on $\widetilde{\mathbb{P}}^N$ which is contracted by f', i.e.

$\phi(C)(=\pi'(\widetilde{\phi}(\widetilde{C})))$ is a line on $\mathbb{P}^N$ containing q. Note that ϕ is étale onto the image at p, and $\phi(C)$ is smooth at $q=\phi(p)$, so C is also smooth at p. Since $f(\widetilde{C})$ is a point, $\widetilde{C}.(\pi^*L-\sum_{i=1}^d E_i)=0$ holds and $\widetilde{C}\cap E_i\subset\widetilde{Z}\cap E_i=\emptyset$ for $i\neq 1$. Thus $0=\widetilde{C}.(\pi^*L-E_1)=C.L-m_p(C)$ holds. From this, we have

$$1=m_p(C)=C.L=\phi_*(C).\mathcal{O}_{\mathbb{P}^N}(1)=\deg(\phi|_C:C\to\phi(C)).$$

Thus $\phi|_C:C\to\phi(C)\cong\mathbb{P}^1$ is an isomorphism and $C.L=1$, i.e. C is a line on X containing p. So (X,L) is covered by lines. □

Remark 5.3. The assumption that $|L|$ is base point free is necessary in Proposition 5.2. For example, let (S,L) be a non-rational polarized smooth surface such that $L^2=1$, e.g. S is a Godeaux surface and L is the canonical divisor K_S. Then $\varepsilon(S,L;1)=1$ holds by [EL] and the assumption $L^2=1$. But S is not covered by lines because S is non-rational.

By Theorem 1.1 and Proposition 5.2 , we obtain the following:

Corollary 5.4 (=Theorem 1.2). *Let $P\subset\mathbb{R}^n$ be a lattice polytope of dimension n. Then the following are equivalent:*

i) *P has lattice width one,*
ii) *(X_P,L_P) is covered by lines,*
iii) *$\varepsilon(X_P,L_P;1)=1$.*

Proof. i) $\Leftrightarrow$ ii) follows from Theorem 1.1 and Remark 2.3. ii) $\Leftrightarrow$ iii) follows from Proposition 5.2. □

As stated in Introduction, this corollary tells us for which P the Seshadri constant $\varepsilon(X_P,L_P;1)$ is one. We note that Nakamaye [Na] gives an explicit description for which polarized abelian variety (A,L) the Seshadri constant $\varepsilon(A,L;1)$ is one.

References

[BN] V. Batyrev and B. Nill, *Combinatorial aspects of mirror symmetry*, Integer points in polyhedra-geometry, number theory, representation theory, algebra, optimization, statistics, 35-66, Contemp. Math., 452, Amer. Math. Soc., Providence, RI, 2008.

[CD] C. Casagrande and S. Di Rocco, *Projective $\mathbb{Q}$-factorial toric varieties covered by lines*, Commun. Contemp. Math. 10 (2008), no. 3, 363-389.

[CC] R. Curran and E. Cattani, *Restriction of A-discriminants and dual defect toric varieties*, J. Symbolic Comput. 42 (2007), no. 1-2, 115-135.

[DDP] A. Dickenstein, S. Di Rocco, and R. Piene, *Classifying smooth lattice polytopes via toric fibrations*, Adv. Math. 222 (2009), no. 1, 240-254.

[DN] A. Dickentein and B. Nill, *A simple combinatorial criterion for projective toric manifolds with dual defect*, Math. Res. Lett. 17 (2010), no. 3, 435-448.

[DHNP] S. Di Rocco, C. Haase, B. Nill, and A. Paffenholz, *Polyhedral adjunction theory*, arXiv:1105.2415.

[EL] L. Ein and R. Lazarsfeld, *Seshadri constants on smooth surfaces*, Journees de Geometrie Algebrique d'Orsay (Orsay, 1992). Asterisque No. 218 (1993), 177-186.

[Ei] D. Eisenbud, *Commutative Algebra,with a View Toward Algebraic Geometry*, Graduate Texts in Math, no.150, Springer-Verlag, New York, 1995.

[Es] A. Esterov, *Newton polyhedra of discriminants of projections*, Discrete Comput. Geom. 44 (2010), no. 1, 96-148.

[Fu] W. Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, 131. The William H. Roever Lectures in Geometry. Princeton University Press, Princeton, NJ, 1993.

[GKZ] I. Gelfand, M. Kapranov, and A. Zelevinsky, *Discriminants, resultants, and multidimensional determinants*, Mathematics: Theory & Applications. Birkhuser Boston, Inc., Boston, MA, 1994. x+523 pp.

[La] R. Lazarsfeld, *Positivity in algebraic geometry I*, Ergebnisse der Mathematik undihrer Grenzgebiete, vol. 48. Springer, Berlin (2004).

[Na] M. Nakamaye, *Seshadri constants on abelian varieties*, Amer. J. Math. 118 (1996), no. 3, 621-635.

[Te] E.A. Tevelev, *Projective duality and homogeneous spaces*, Encyclopaedia of Mathematical Sciences, 133. Invariant Theory and Algebraic Transformation Groups, IV. Springer-Verlag, Berlin, 2005.

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